# MULTIPLE INTEGRALS - TASKS ( III PART)

## Calculate the area of the plane by a double integral

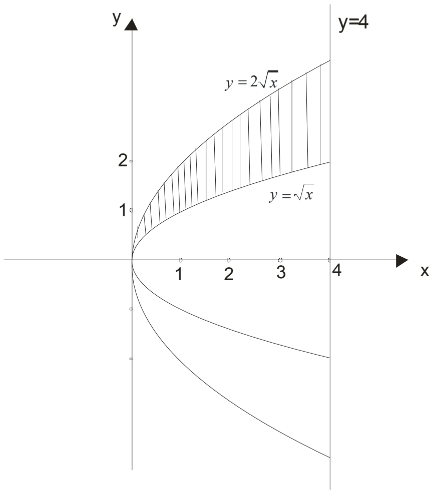
**Area $D$ in the plane $xOy$ can be found using the formula:** $$P = \iint_D dxdy$$

**Example 1.**

Calculate area which contain the following lines: $y=\sqrt{x}$, $y=2\sqrt{x}$ and $y = 4$.

*Solution:*

First, we will, as always, draw a picture and set boundaries.




The area of integration is shaded in picture $D:\begin{cases} 0 \le x \le 4 \\ \sqrt{x} \le y \le 2\sqrt{x} \end{cases}$

Using the above formula, we calculate :

$$P = \iint_D dxdy = \int_0^4 \left( \int_{\sqrt{x}}^{2\sqrt{x}} dy \right) dx = \int_0^4 \left( y \Big|_{\sqrt{x}}^{2\sqrt{x}} \right) dx = \int_0^4 \left( 2\sqrt{x} - \sqrt{x} \right) dx = \int_0^4 \sqrt{x} dx =$$

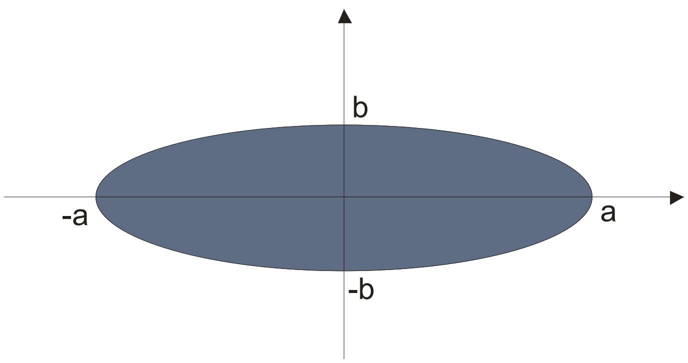
$$= \int_0^4 x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^4 = \frac{2}{3}\left( 4^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) = \frac{2}{3} \cdot 8 = \boxed{\frac{16}{3}}$$

**Example 2.**

Calculate area limited with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

*Solution:*

Naravno, ovde je u pitanju elipsa. Mi treba da izračunamo površinu unutar nje...




Of course, it is ellipse.

Here is convenient to take the so-called **elliptic coordinates:**

x = $a\ r\ cos\,\varphi$

y = $b\ r\ sin\ \varphi$ Then is : $\iint_D z(x,y)dxdy = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(ar\cos\varphi, br\sin\varphi)abrdr$

$|J| = abr$

To see why these **elliptic coordinates** is good:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{(ar\cos\varphi)^2}{a^2}+\frac{(br\sin\varphi)^2}{b^2}=1$$
$$\frac{\cancel{a^2}r^2\cos^2\varphi}{\cancel{a^2}}+\frac{\cancel{b^2}r^2\sin^2\varphi}{\cancel{b^2}}=1$$
$$r^2\cos^2\varphi+r^2\sin^2\varphi=1$$
$$r^2(\cos^2\varphi+\sin^2\varphi)=1$$
$$r^2=1\rightarrow r=1$$

So: $0\le r\le 1$ , since the angle takes the whole circuit, it is $0\le\varphi\le 2\pi$ .

Area D : $\begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$

Now solve the double integral :

$$P = \iint_D dxdy = \int_0^{2\pi} d\varphi \int_0^1 abrdr = ab\int_0^{2\pi} \left(\frac{r^2}{2}\right)\Bigg|_0^1 d\varphi = \frac{1}{2}ab\int_0^{2\pi} d\varphi = \frac{1}{2}ab \cdot 2\pi = \boxed{ab\pi}$$

Area of ellipse is then calculated by the formula P = $ab\pi$

**Example 3.**

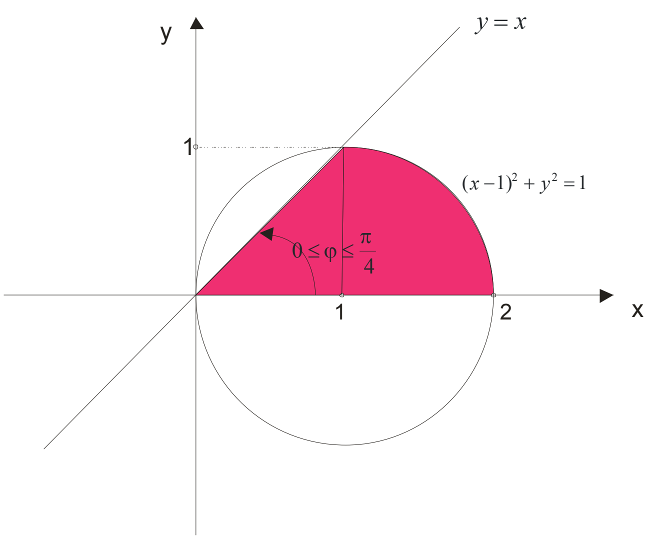
Calculate area limited with: $x^2 + y^2 = 2x$ , $y = x$ and $y = 0$

*Solution:*

Circle:

$$x^2 + y^2 - 2x = 0$$
$$x^2 - 2x + 1 - 1 + y^2 = 0$$
$$(x-1)^2 + y^2 = 1$$

The sections were evident at x = 0 and x = 1




We will use polar coordinates : $x = r\cos\varphi$, $y = r\sin\varphi$, $|J| = r$

$$x^2+y^2-2x=0$$
$$(r\cos\varphi)^2+(r\sin\varphi)^2-2r\cos\varphi=0$$
$$r^2(\cos^2\varphi+\sin^2\varphi)=2r\cos\varphi$$
$$r^2=2r\cos\varphi\to r=2\cos\varphi$$

So: $0\le r\le 2\cos\varphi$

The angle goes from line y = 0 to line y = x . So: $0\le\varphi\le\dfrac{\pi}{4}$

Now we can calculate the required area:

$$P=\iint_D dxdy=\int_0^{\frac{\pi}{4}}d\varphi\int_0^{2\cos\varphi}rdr=\int_0^{\frac{\pi}{4}}\left(\frac{r^2}{2}\right)\Bigg|_0^{2\cos\varphi}d\varphi=\int_0^{\frac{\pi}{4}}\left(\frac{4\cos^2\varphi}{2}\right)d\varphi=$$
$$=2\int_0^{\frac{\pi}{4}}\cos^2\varphi d\varphi$$

trigonometric formula : $\cos^2\varphi=\dfrac{1+\cos2\varphi}{2}$

$$P=\iint_D dxdy=2\int_0^{\frac{\pi}{4}}\cos^2\varphi d\varphi=\cancel{2}\int_0^{\frac{\pi}{4}}\frac{1+\cos2\varphi}{\cancel{2}}d\varphi=\int_0^{\frac{\pi}{4}}(1+\cos2\varphi)d\varphi=$$
$$=\left(\varphi+\frac{1}{2}\sin2\varphi\right)\Bigg|_0^{\frac{\pi}{4}}=\left(\frac{\pi}{4}+\frac{1}{2}\sin2\cdot\frac{\pi}{4}\right)-\left(0+\frac{1}{2}\sin2\cdot0\right)=\boxed{\frac{\pi}{4}+\frac{1}{2}}$$

So far we have used the polar and cylindrical coordinates.

But in more serious tasks we have to use so-called **generalized polar coordinates** $r$ and $\varphi$ by formulas:

$$\left.\begin{array}{l}x=ar\cos^\alpha\varphi\\ y=br\sin^\alpha\varphi\end{array}\right\}\to|J|=\alpha\cdot abr\cdot\cos^{\alpha-1}\varphi\cdot\sin^{\alpha-1}\varphi$$

Value for $\alpha$ is taken depending on the specific situation .

**Example 4 .**

Calculate the area limited with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{h}+\frac{y}{k}$ If the parameters $a,b,$ $h$ and $k$ are positive.

*Solution:*

As is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{h}+\frac{y}{k}$$
$$\frac{x^2}{a^2}-\frac{x}{h}+\frac{y^2}{b^2}-\frac{y}{k}=0$$
$$\boxed{\frac{x^2}{a^2}-\frac{x}{h}+\left(\frac{a}{2h}\right)^2}-\left(\frac{a}{2h}\right)^2+\boxed{\frac{y^2}{b^2}-\frac{y}{k}+\left(\frac{b}{2k}\right)^2}-\left(\frac{b}{2k}\right)^2=0$$
$$\left(\frac{x}{a}-\frac{a}{2h}\right)^2+\left(\frac{y}{b}-\frac{b}{2k}\right)^2=\left(\frac{a}{2h}\right)^2+\left(\frac{b}{2k}\right)^2$$
$$\left(\frac{x}{a}-\frac{a}{2h}\right)^2+\left(\frac{y}{b}-\frac{b}{2k}\right)^2=\frac{a^2}{4h^2}+\frac{b^2}{4k^2}$$

Now think about that. Convenient would be to destroy terms in the brackets. So we'll take that:

$$\frac{x}{a}-\frac{a}{2h}=r\cos\varphi\rightarrow\frac{x}{a}=r\cos\varphi+\frac{a}{2h}\rightarrow\boxed{x=ar\cos\varphi+\frac{a^2}{2h}}$$

$$\frac{y}{b}-\frac{b}{2k}=r\sin\varphi\rightarrow\frac{y}{b}=r\sin\varphi+\frac{b}{2k}\rightarrow\boxed{y=br\sin\varphi+\frac{b^2}{2k}}$$

So:

$$\left.\begin{aligned}x&=ar\cos\varphi+\frac{a^2}{2h}\\ y&=br\sin\varphi+\frac{b^2}{2k}\end{aligned}\right\}\rightarrow|J|=abr$$

Now to define the borders.

$$\left(\frac{x}{a}-\frac{a}{2h}\right)^2+\left(\frac{y}{b}-\frac{b}{2k}\right)^2=\frac{a^2}{4h^2}+\frac{b^2}{4k^2}$$

All this gives $r^2$

$$r^2=\frac{a^2}{4h^2}+\frac{b^2}{4k^2}$$

$$r^2=\frac{1}{4}\left(\frac{a^2}{h^2}+\frac{b^2}{k^2}\right)\to r=\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}$$

So: $0\le r\le\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}$

And $0\le\varphi\le 2\pi$

Now calculate the required area:

$$P=\iint_D dxdy=\int_0^{2\pi}d\varphi\int_0^{\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}}abrdr=ab\int_0^{2\pi}d\varphi\int_0^{\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}}rdr$$

We can write that $\int_0^{2\pi}d\varphi=2\pi$,

Further we have:

$$P=\iint_D dxdy=\int_0^{2\pi}d\varphi\int_0^{\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}}abrdr=2ab\pi\int_0^{\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}}rdr=$$

$$=\cancel{2}ab\pi\left(\frac{r^2}{\cancel{2}}\right)\Big|_0^{\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}}=ab\pi\left(\frac{1}{2}\sqrt{\frac{a^2}{h^2}+\frac{b^2}{k^2}}\right)^2=ab\pi\frac{1}{4}\left(\frac{a^2}{h^2}+\frac{b^2}{k^2}\right)=\boxed{\frac{ab\pi}{4}\left(\frac{a^2}{h^2}+\frac{b^2}{k^2}\right)}$$

**Example 5.**

Calculate the area limited with:

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$$
$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=4$$
$$\frac{x}{a}=\frac{y}{b}$$
$$8\frac{x}{a}=\frac{y}{b}$$
$$x>0, y>0$$

<u>*Solution:*</u>

We will use **the generalized polar coordinates:** $\left.\begin{array}{l} x=ar\cos^{\alpha}\varphi \\ y=br\sin^{\alpha}\varphi \end{array}\right\}\rightarrow|J|=\alpha\cdot abr\cdot\cos^{\alpha-1}\varphi\cdot\sin^{\alpha-1}\varphi$

$$\left.\begin{array}{l} x=ar\cos^{3}\varphi \\ y=br\sin^{3}\varphi \end{array}\right\}\rightarrow|J|=3\cdot abr\cdot\cos^{3-1}\varphi\cdot\sin^{3-1}\varphi\rightarrow\boxed{|J|=3\cdot abr\cdot\cos^{2}\varphi\cdot\sin^{2}\varphi}$$

Whay $\alpha=3$ ?

$\left.\begin{array}{l} x=ar\cos^{3}\varphi \\ y=br\sin^{3}\varphi \end{array}\right\}$ replace in $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ and we have :

$$\left(\frac{ar\cos^{3}\varphi}{a}\right)^{\frac{2}{3}}+\left(\frac{br\sin^{3}\varphi}{b}\right)^{\frac{2}{3}}=1$$
$$r^{\frac{2}{3}}\cos^{2}\varphi+r^{\frac{2}{3}}\sin^{2}\varphi=1$$
$$r^{\frac{2}{3}}=1\rightarrow\boxed{r=1}$$

Next $\left.\begin{array}{l} x=ar\cos^{3}\varphi \\ y=br\sin^{3}\varphi \end{array}\right\}$ replace in $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=4$ and we have :

$$\left(\frac{\cancel{a} r\cos^3\varphi}{\cancel{a}}\right)^{\frac{2}{3}}+\left(\frac{\cancel{b} r\sin^3\varphi}{\cancel{b}}\right)^{\frac{2}{3}}=4$$

$$r^{\frac{2}{3}}\cos^2\varphi+r^{\frac{2}{3}}\sin^2\varphi=4$$

$$r^{\frac{2}{3}}=4\rightarrow\boxed{r=8}$$

So: $1\le r\le 8$

Now to define the limits for the angle:

$$\frac{x}{a}=\frac{y}{b}$$

$$\frac{ar\cos^3\varphi}{a}=\frac{br\sin^3\varphi}{b}$$

$$\cos^3\varphi=\sin^3\varphi\rightarrow\frac{\sin^3\varphi}{\cos^3\varphi}=1\rightarrow tg^3\varphi=1\rightarrow\boxed{\varphi=arctg1}$$

Still have:

$$8\frac{x}{a}=\frac{y}{b}$$

$$8\frac{ar\cos^3\varphi}{a}=\frac{br\sin^3\varphi}{b}$$

$$8\cos^3\varphi=\sin^3\varphi\rightarrow\frac{\sin^3\varphi}{\cos^3\varphi}=8\rightarrow tg^3\varphi=2^3\rightarrow\boxed{\varphi=arctg2}$$

$$So: \quad arctg1\le\varphi\le arctg2$$

Now calculate the required area:

$$P=\iint_D dxdy=\int_{arctg1}^{arctg2}d\varphi\int_1^8 3\cdot abr\cdot\cos^2\varphi\cdot\sin^2\varphi dr=3ab\int_{arctg1}^{arctg2}\cos^2\varphi\cdot\sin^2\varphi d\varphi\int_1^8 rdr=$$

As is $\int_1^8 rdr=\frac{r^2}{2}\Big|_1^8=\frac{64}{2}-\frac{1}{2}=\frac{63}{2}$, we have

$$=\frac{63}{2}\cdot 3ab\int_{arctg1}^{arctg2}\cos^2\varphi\cdot\sin^2\varphi d\varphi$$

This integral will be most easily solved using a formula from trigonometry:

$$\cos^2\varphi\cdot\sin^2\varphi=\frac{4}{4}\cos^2\varphi\cdot\sin^2\varphi=\frac{\sin^2 2\varphi}{4}=\frac{1}{4}\sin^2 2\varphi=\frac{1}{4}\left(\frac{1-\cos 4\varphi}{2}\right)=\frac{1}{8}\left(1-\cos 4\varphi\right)$$

Now, we have :

$$P=\frac{63}{2}\cdot 3ab\int_{arctg1}^{arctg2}\cos^2\varphi\cdot\sin^2\varphi d\varphi=\frac{189}{16}ab\int_{arctg1}^{arctg2}(1-\cos 4\varphi)d\varphi$$

Solution:

$$\boxed{P=\frac{189}{16}ab\cdot(arctg\frac{1}{3}+\frac{6}{25})}$$

**Example 6 .**

Calculate the area limited with:

$x^2=ay$

$x^2=by$

$x^3=cy^2$

$x^3=dy^2$

$(0<a<b)\wedge(0<c<d)$

<u>*Solution:*</u>

Here is convenient to take replacement $u$ and $v$.

But how to choose?

Let's look at the first two equations:

$$x^2=ay\rightarrow\frac{x^2}{y}=a$$

$$x^2=by\rightarrow\frac{x^2}{y}=b$$

We take $\boxed{u=\frac{x^2}{y}}$

From the other two we have:

$$x^3 = cy^2 \to \frac{x^3}{y^2} = c$$

$$x^3 = dy^2 \to \frac{x^3}{y^2} = d$$

It is convenient to take : $\boxed{v = \frac{x^3}{y^2}}$

Replacements are :

$$u = \frac{x^2}{y}$$

$$v = \frac{x^3}{y^2}$$

From here, we express x and y:

$$u = \frac{x^2}{y} \to y = \frac{x^2}{u}$$

$$v = \frac{x^3}{y^2} \to x^3 = y^2 v \to x^3 = \left(\frac{x^2}{u}\right)^2 \cdot v \to x^3 = \frac{x^4}{u^2} v \to \boxed{x = \frac{u^2}{v}}$$

$$y = \frac{x^2}{u} \to y = \frac{\left(\frac{u^2}{v}\right)^2}{u} \to \boxed{y = \frac{u^3}{v^2}}$$

We are looking for Jacobian:

$$\left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ \frac{3u^2}{v^2} & -\frac{2u^3}{v^3} \end{vmatrix} = \left| -\frac{4u^4}{v^4} + \frac{3u^4}{v^4} \right| = \left| -\frac{u^4}{v^4} \right| = \frac{u^4}{v^4}$$

To determine the boundaries:

$$\left.\begin{array}{l} u=\dfrac{x^2}{y}=a \\ u=\dfrac{x^2}{y}=b \end{array}\right\} \to \boxed{a \le u \le b}$$

$$\left.\begin{array}{l} v=\dfrac{x^3}{y^2}=c \\ v=\dfrac{x^3}{y^2}=d \end{array}\right\} \to \boxed{c \le v \le d}$$

Now we can calculate the area:

$$P=\iint_D dxdy=\int_a^b du\int_c^d \frac{u^4}{v^4}dv=\int_a^b u^4du\int_c^d \frac{1}{v^4}dv$$

These two integrals is not difficult to solve and get:

$$P=\frac{1}{15}\left(b^5-a^5\right)\left(\frac{1}{c^3}-\frac{1}{d^3}\right)$$